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Goldstone modes and coexistence in isotropic N-vector models

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Abstract. Coexistence in the d-dimensional, isotropic, N-component ϕ^4 model is studied at first order in $\varepsilon = 4 - d$, using a renormalisation technique similar to that appropriate for bicritical crossover. The central result is the identification of a zero-temperature fixed point controlling the Goldstone singularities, with coupling constant $u^{**} \propto \varepsilon/(N-1) + O(\varepsilon^2)$, at which the transverse fields have canonical scaling dimensions. Both critical and Goldstone singularities are fully regularised by our renormalisation group, and a complete exponentiation of the equation of state is achieved. The leading behaviour of the correlation functions is also exhibited.

1. Introduction

The question of coexistence below the critical temperature, T_c, of an isotropic N-component system has attracted considerable theoretical interest over the years (see e.g. Brézin and Wallace 1973, Wallace and Zia 1975, Nelson 1976, Brézin and Zinn-Justin 1976, Schäfer and Horner 1978, and references given by these authors). Central to these investigations is the existence of $(N-1)$ massless Goldstone modes associated with fluctuations transverse to the direction of spontaneous ordering. As is well known, these spin-wave-like fluctuations induce infrared singularities for all $T < T_c$: at the mean field level there is an infinite transverse susceptibility, and analysis of fluctuation corrections reveals, for spatial dimensionalities $d < 4$, singularities in other thermodynamic functions, notably the longitudinal correlation function, which is predicted to diverge as $p^{-\varepsilon}$ for small wavevectors.

In one sense, the most appropriate field theory models for studying this behaviour are the nonlinear σ model (Brézin and Zinn-Justin 1976) and its variants (see e.g. Brézin and Wallace 1973), which focus attention directly on the transverse modes. Evidence from these sources indicates fairly conclusively that the Goldstone singularities are governed by the canonical dimensions of the fluctuating fields. In particular, Brézin and Zinn-Justin (1976) have identified a trivial, infrared stable fixed point of the renormalisation group which controls coexistence. The same analysis also yields, for $N > 2$, a non-trivial fixed point, which is infrared unstable in the temperature variable, and controls the critical singularity. Thus, for this model, one has a scheme which correctly incorporates both critical and coexistence singularities, and may be applied in the whole critical region. A drawback of this scheme is that it is limited in practice to dimensionalities close to the lower critical dimensionality $d_l = 2$. This means that Ising-like ($N = 1$) and XY-like ($N = 2$) systems must be treated as special cases (in particular, the Ising case has no Goldstone modes and $d_l = 1$.) Furthermore, it is not easy to see in detail the relation between this scheme and those based on the ϕ^4 model which, according to general universality arguments, purports to describe the infrared behaviour of the same systems. Here, by contrast, one studies most easily dimensionalities near four, where the critical behaviour of the Ising and XY models and coexistence in the XY model exhibit no exceptional features.

Attempts to incorporate Goldstone singularities into the ε expansion about four dimensions have followed two routes. Firstly, one can obtain certain information about thermodynamic quantities by *assuming* canonical behaviour for the transverse modes and treating by hand those contributions to the perturbation series which are directly affected (Wallace and Zia 1975, Mazenko 1976). Secondly, exponentiations of the equation of state and the correlation functions have been obtained by Nelson (1976) and by Schäfer and Horner (1978). These authors combine renormalisation group treatment of the critical singularity with resummation procedures of a partly conjectural nature for the Goldstone modes. The different resummations used in these two works give different results: that of Schäfer and Horner would appear to be the more reliable. Such an exponentiation has also been achieved by Nicoll and Chang (1978, see also Nicoll 1980) by means of nonlinear solution of a rather general form of renormalisation group equation. In fact, their results are similar to those presented here. Their method bypasses any detailed consideration of the role played by the Goldstone modes (though this is by no means a deficiency).

In the present work, we study coexistence in the ϕ^4 model, using an ε expansion about four dimensions. We exhibit a renormalisation scheme which explicitly takes account of both critical and Goldstone singularities, and thus yields an exponentiation of the equation of state which avoids those ambiguities which are not inherent in the ε expansion. Our explicit first-order results do not seem to contain any essentially new information, though the technique we use and the results we obtain have the merit of relative simplicity. (The same claim, however, would no doubt be made by other authors for their favourite techniques!) The novel feature of our work is the identification of a zero-temperature fixed point which controls the coexistence behaviour. We believe that this result contributes to elucidating the presumed equivalence of the ϕ^4 and nonlinear σ models. In common with several earlier studies, we obtain an equation of state which is exact in the spherical model limit, $N\to\infty$.

Our renormalisation scheme is based on the magnetisation–temperature phase diagram, in which the critical point appears as a kind of bicritical point; the boundaries of the coexistence region, at which the Goldstone singularities occur, then play a role analogous to that of the critical loci which terminate at a real bicritical point. This idea is implicit in several of the works we have cited, but does not seem to have been fully exploited. We can thus use a renormalisation prescription analogous to that introduced by Amit and Goldschmidt (1978) to study bicritical crossover. This scheme is described in § 2. In § 3 we demonstrate the Gaussian character of the fixed point which controls the coexistence singularity, and in § 4 we use our extended renormalisation group to obtain to $O(\varepsilon)$ a fully exponentiated form of the equation of state, together with the longitudinal and transverse correlation functions. A proof that this scheme works is outlined in an Appendix.

2. Renormalisation

Our starting point is the usual $O(N)$ symmetric Landau–Ginzburg–Wilson model, defined by the Hamiltonian density

$$\mathcal{H}(x)=\tfrac{1}{2}|\nabla \boldsymbol{\phi}_0|^2+\tfrac{1}{2}r_0|\boldsymbol{\phi}_0|^2+(1/4!)u_0|\boldsymbol{\phi}_0|^4, \tag{2.1}$$

where $\boldsymbol{\phi}_0(x)$ is an N-component field, u_0 may be taken as constant in the critical region, and r_0 may be identified as

$$r_0 \approx r_{0c}(u_0)+\text{constant}(T-T_c)/T. \tag{2.2}$$

The critical behaviour of this model is most economically obtained by use of the minimal subtraction procedure of t'Hooft and Veltman (1972; see also Lawrie 1976, Amit 1978), whereby one defines renormalised quantities $\boldsymbol{\phi}(x)$, t, $\tilde{u}$ in such a way as to remove poles at $\varepsilon=0$ from the renormalised, one-particle-irreducible vertex functions $\Gamma^{(2)}$, $\partial\Gamma^{(2)}/\partial p^2$ and $\Gamma^{(4)}$. At one-loop order, the result is

$$\boldsymbol{\phi}_0(x)=\boldsymbol{\phi}(x)[1+\mathrm{O}(\tilde{u}^2)], \tag{2.3}$$

$$r_0=r_{0c}+\kappa^2 t\{1+[(N+2)/6\varepsilon]S\tilde{u}+\mathrm{O}(\tilde{u}^2)\}, \tag{2.4}$$

$$u_0=\kappa^{\varepsilon}\tilde{u}\{1+[(N+8)/6\varepsilon]S\tilde{u}+\mathrm{O}(\tilde{u}^2)\}, \tag{2.5}$$

where κ is an arbitrary parameter with the dimensions of inverse length, ensuring that $t\propto(T-T_c)/T$ and $\tilde{u}$ are dimensionless, and $S=2\pi^{d/2}/(2\pi)^d(\frac{1}{2}d-1)!$. In view of (2.3), we shall henceforth drop the distinction between bare and renormalised fields.

Near the coexistence curve, this renormalisation is inadequate to exponentiate the singularities induced by the Goldstone modes. We achieve this exponentiation by an extension of (2.3)–(2.5) analogous to that introduced by Amit and Goldschmidt (1978) to study crossover effects near a bicritical point. In order to focus on behaviour *at* the coexistence curve, we first define new fields, $\sigma(x)$ and $\boldsymbol{\pi}(x)$, according to

$$\boldsymbol{\phi}=[\sigma+(3/u_0)^{1/2}m_0,\ \boldsymbol{\pi}] \tag{2.6}$$

where the transverse field $\boldsymbol{\pi}(x)$ has $(N-1)$ components, and $m_0(r_0, u_0)$ is chosen so that σ has zero expectation value: it is proportional to the spontaneous magnetisation. One may identify

$$m_0^2=-2r_0+2A \tag{2.7}$$

where the counterterm

$$A=-\tfrac{1}{2}u_0\int\frac{\mathrm{d}^dq}{q^2+m_0^2}-\frac{(N-1)}{6}u_0\int\frac{\mathrm{d}^dq}{q^2}+\mathrm{O}(u_0^2) \tag{2.8}$$

ensures both that σ has zero expectation value and, via a Ward identity, that $\boldsymbol{\pi}$ remains massless to all orders of perturbation theory. Up to an unimportant constant, the Hamiltonian density now reads

$$\mathcal{H}=\tfrac{1}{2}|\nabla\boldsymbol{\pi}|^2+\tfrac{1}{2}|\nabla\sigma|^2+\tfrac{1}{2}m_0^2\sigma^2+\tfrac{1}{6}(3u_0)^{1/2}m_0(\pi^2+\sigma^2)\sigma$$
$$+(1/4!)u_0(\pi^2+\sigma^2)^2+\tfrac{1}{2}A(\pi^2+\sigma^2)+(3/u_0)^{1/2}m_0A\sigma. \tag{2.9}$$

We now wish to define dimensionless renormalised parameters m and u in such a way that solution of the associated renormalisation group equation correctly exponentiates the Goldstone mode singularities. To motivate our prescription, consider the longitudinal two-point vertex function, $\Gamma_{\sigma\sigma}$, and define $f_{\sigma\sigma}$ by

$$\Gamma_{\sigma\sigma}(\kappa^2p^2;\ m, u, \kappa)=\kappa^2m^2f_{\sigma\sigma}(p^2;\ m, u), \tag{2.10}$$

where the wavevector $\boldsymbol{p}$ has been rendered dimensionless by extraction of a factor of κ. We anticipate that solution of the renormalisation group equation will yield, in the limit $p \to 0$, a relation of the form

$$f_{\sigma\sigma}(p^2; m, u) \approx |p|^a f_{\sigma\sigma}(1; m/|p|^b, u^{**}), \tag{2.11}$$

where $a = O(\varepsilon)$, $b = 1 + O(\varepsilon)$ and u^{**} is a suitable fixed point value of u. Clearly, the exponent a will represent the singularity correctly, provided that $f_{\sigma\sigma}$ on the right of (2.11) remains finite and non-zero in the limit $m/p^b \to \infty$. To ensure this, we propose to define m and u in such a way that $f_{\sigma\sigma}(p^2; m, u)$ remains finite and non-zero when $m \to \infty$, with $p^2 \neq 0$. In fact, as we show in the Appendix, one can ensure similar behaviour, at all orders of perturbation theory, for each of the functions

$$f_{\sigma^n} = \kappa^{-d_n} m^{-n} \Gamma_{\sigma^n}, \tag{2.12}$$

$$f_{\pi\pi\sigma^n} = \kappa^{-d_{n+2}} m^{-n} \Gamma_{\pi\pi\sigma^n}, \tag{2.13}$$

associated with the correlations of n longitudinal and zero or two transverse fields at distinct points in space, where $d_n = d + n(1 - \frac{1}{2}d)$ is the canonical dimension of the vertex function.

At the order of one-loop diagrams, which we consider explicitly, this renormalisation allows us to obtain, for arbitrary magnetisation, the two-point correlation functions and, as a special case, the equation of state to order ε. At higher orders, a new wavefunction renormalisation, ensuring the finiteness of $\partial f_{\sigma\sigma}/\partial p^2$ as $m \to \infty$, would presumably be required to complete the prescription. Specifically, then, we define u and m by requiring that $f_{\sigma\sigma}$ and $f_{\pi\pi\sigma}$ contain no poles at $\varepsilon = 0$, and remain finite when $m \to \infty$ at fixed, non-zero wavevector, the latter condition being applied order by order in the double expansion in powers of u and ε. The calculation is very similar to that described in detail by Amit and Goldschmidt (1978), and we merely quote the results

$$u_0 = \kappa^{\varepsilon} u\{1 + [(N+8)/6\varepsilon]Su - \tfrac{3}{4}Su \ln(\alpha + m^2) + O(u^2)\}, \tag{2.14}$$

$$m_0^2 = \kappa^2 m^2\{1 + [(N+8)/6\varepsilon]Su - \tfrac{3}{4}Su \ln(\alpha + m^2) + O(u^2)\}. \tag{2.15}$$

In these expressions, α is an arbitrary constant, since only the large-m behaviour is important for the coexistence singularity. However, for the purpose of studying the crossover to ordinary critical behaviour, the choice

$$\alpha = 3 \tag{2.16}$$

turns out to be particularly convenient and, as we shall see, facilitates a complete exponentiation of the equation of state at this order.

The desired renormalisation group equation is obtained in the usual way, by applying to the vertex functions the differential operator

$$\kappa \frac{\partial}{\partial \kappa}\bigg|_0 = \kappa \frac{\partial}{\partial \kappa} + W \frac{\partial}{\partial u} - \gamma_m m^2 \frac{\partial}{\partial m^2} \tag{2.17}$$

where the derivative on the left is at fixed m_0 and u_0. Using (2.12) and (2.13) and ignoring wavefunction renormalisation at this order, we obtain

$$\left(-\lambda \frac{\partial}{\partial \lambda} + W \frac{\partial}{\partial u} - \gamma_m m^2 \frac{\partial}{\partial m^2} + d_{n+l} - \tfrac{1}{2} l \gamma_m\right) f_{\pi^n\sigma^l}(\lambda p_1 \ldots \lambda p_{n+l}; m, u) = 0, \tag{2.18}$$

the λ derivative arising from the rescaling of the wavevector in (2.10). Application of the same operator to (2.14) and (2.15) gives the coefficients as

$$W=-\varepsilon u+\left(\frac{N+8}{6}-\frac{3}{2}\frac{m^2}{(3+m^2)}\right)Su^2+O(\varepsilon u^2, u^3), \tag{2.19}$$

$$\gamma_m=2-\left(\frac{N+8}{6}-\frac{3}{2}\frac{m^2}{(3+m^2)}\right)Su+O(\varepsilon u, u^2). \tag{2.20}$$

Finally, to study correlation functions away from the coexistence curve, one must add a magnetic field term, $-H\sigma$, to the Hamiltonian density. The new vertex functions are naturally expressed in terms of the dimensionless increment in magnetisation,

$$\Delta M=\kappa^{\varepsilon/2-1}\langle\sigma\rangle. \tag{2.21}$$

They are given by

$$\Gamma_{\pi^n\sigma^l}(p_1\ldots p_{n+l}; m, u, \Delta M, \kappa)$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}(\kappa^{1-\varepsilon/2}\Delta M)^k\Gamma_{\pi^n\sigma^{l+k}}(p_1\ldots p_{n+l}, 0\ldots 0; m, u, \kappa) \tag{2.22}$$

or

$$f_{\pi^n\sigma^l}=\sum_{k=0}^{\infty}\frac{1}{k!}(m\Delta M)^k f_{\pi^n\sigma^{l+k}}. \tag{2.23}$$

Near the coexistence curve, (2.23) remains finite in the limit $m\to\infty$, with $(m\Delta M)$ fixed. Consequently, it is appropriate to write the new renormalisation group equation in terms of the parameter

$$z=(\tfrac{1}{12}u)^{1/2}m\Delta M, \tag{2.24}$$

in which the factor $(u/12)^{1/2}$ ensures that the quantity

$$\mu=(\tfrac{1}{2}m^2)^{1/2}+z(\tfrac{1}{2}m^2)^{-1/2} \tag{2.25}$$

is proportional to the total magnetisation. One obtains

$$\left(-\lambda\frac{\partial}{\partial\lambda}+W\frac{\partial}{\partial u}-\gamma_m m^2\frac{\partial}{\partial m^2}-\gamma_z z\frac{\partial}{\partial z}+d_{n+l}-\tfrac{1}{2}l\gamma_m\right)f_{\pi^n\sigma^l}(\lambda p_i; m, u, z)=0 \tag{2.26}$$

where, at this order, $\gamma_z=\gamma_m$.

3. The coexistence fixed point

In order to study the asymptotic coexistence singularity, we require an approximate solution of the renormalisation group equation (2.26) in the large-m limit. In this limit, inspection of (2.19) shows that there is an infrared stable fixed point

$$Su^{**}=6\varepsilon/(N-1)+O(\epsilon^2). \tag{3.1}$$

Furthermore, the limit $m\to\infty$ corresponds to $r_0\to-\infty$, and hence, via (2.2), to $T=0$. At this coexistence fixed point, we obtain

$$\gamma_m=\gamma_z=2-\varepsilon. \tag{3.2}$$

Now, despite the apparently non-trivial value (3.1), the fixed point Hamiltonian is essentially Gaussian in character, and therefore (3.2) should, in fact, be an exact result. To see this, observe that the functions $f_{\pi^l\sigma^n}$, which remain finite in the limit of interest, are associated with correlations of the fields $\boldsymbol{\pi}(x)$ and

$$s(x)=m\sigma(x). \tag{3.3}$$

If we ignore the distinction between renormalised and bare quantities (a step which is justified *a posteriori* by the Gaussian Hamiltonian we are about to obtain, and more directly by the discussion of the bare theory given in the Appendix), we may substitute (3.3) into the Hamiltonian (2.9) and take the limit $m\to\infty$ to obtain

$$\mathscr{H}=\tfrac{1}{2}|\nabla\boldsymbol{\pi}|^2+\tfrac{1}{2}s^2+\tfrac{1}{6}(3u)^{1/2}|\boldsymbol{\pi}|^2s+(1/4!)u|\boldsymbol{\pi}|^4+\tfrac{1}{2}A|\boldsymbol{\pi}|^2+(3/u)^{1/2}As. \tag{3.4}$$

On shifting the longitudinal field according to

$$\bar{s}=s+\tfrac{1}{6}(3u)^{1/2}|\boldsymbol{\pi}|^2+(3/u)^{1/2}A \tag{3.5}$$

this becomes

$$\mathscr{H}=\text{constant}+\tfrac{1}{2}|\nabla\boldsymbol{\pi}|^2+\tfrac{1}{2}\bar{s}^2, \tag{3.6}$$

which verifies our assertion. Finally, if we add to (3.4) a source term, $-hs$, for the longitudinal field, then (3.6) is modified by the addition of

$$\Delta\mathscr{H}=(3/u)^{1/2}Ah-\tfrac{1}{2}h^2+\tfrac{1}{2}(u/3)^{1/2}h|\boldsymbol{\pi}|^2. \tag{3.7}$$

Thus, below four dimensions, the leading singularities of longitudinal correlation functions are given by the correlations of the operator $|\boldsymbol{\pi}(x)|^2$ (cf Wallace and Zia 1975), whose scaling dimension in the Gaussian ensemble is canonical, and given by (3.2).

4. Thermodynamic functions in the critical region

In § 2 we introduced renormalised parameters m and z appropriate for describing behaviour near the coexistence curve. We now wish to obtain expressions in scaling form for the longitudinal and transverse correlation functions and, as a special case of the latter, the equation of state, which also describe the critical singularities correctly. The critical behaviour is most naturally expressed in terms of the total magnetisation M, and the reduced temperature t, which was defined in (2.4) and is related to m by

$$t=-\tfrac{1}{2}m^2\{1+[\tfrac{1}{2}\ln m^2-\tfrac{3}{4}\ln(3+m^2)]Su+O(u^2)\}. \tag{4.1}$$

In order to interpolate between the two descriptions, we define new variables μ and τ by (2.25) and

$$\tau=-\tfrac{1}{2}m^2\left[1+\frac{1}{2}\ln\left(\frac{m^2}{3+m^2}\right)Su+O(u^2)\right]. \tag{4.2}$$

The relation between μ and M is found to be

$$M^2=\mu^2[1-\tfrac{3}{4}\ln(3+m^2)Su+O(u^2)], \tag{4.3}$$

and we see that $\tau\simeq-\tfrac{1}{2}m^2$ in the limit $m\to\infty$, while near the critical point, $m=0$, the variables μ and τ are respectively proportional to M and t. The renormalisation group equation now takes the form

$$\left(-\lambda\frac{\partial}{\partial\lambda}+W\frac{\partial}{\partial u}-\gamma_\tau\tau\frac{\partial}{\partial\tau}-\gamma_\mu\mu^2\frac{\partial}{\partial\mu^2}+d_{n+l}-\tfrac{1}{2}l\gamma_\mu\right)f_{\pi^n\sigma^l}(\lambda p_i;\tau,\mu,u)=0, \tag{4.4}$$

with

$$W=-\varepsilon u+[\tfrac{1}{6}(N+8)+3\tau/(3-2\tau)]Su^2, \tag{4.5}$$

$$\gamma_\tau=2-[\tfrac{1}{6}(N+2)+\tau/(3-2\tau)]Su, \tag{4.6}$$

$$\gamma_\mu=2-[\tfrac{1}{6}(N+8)+3\tau/(3-2\tau)]Su. \tag{4.7}$$

This equation may be integrated in the usual way, to give the relation

$$f_{\pi^n\sigma^l}(\lambda p_i;\tau,\mu,u)=\lambda^{d_{n+l}}\left(\frac{\bar\mu}{\mu}\right)^l f_{\pi^n\sigma^l}(p_i;\bar\tau,\bar\mu,\bar u) \tag{4.8}$$

where the characteristic functions $\bar u(\lambda)$, $\bar\tau(\lambda)$ and $\bar\mu(\lambda)$ satisfy

$$\lambda\,\partial\bar u/\partial\lambda = W(\bar u,\bar\tau), \tag{4.9}$$

$$\lambda\,\partial\bar\tau/\partial\lambda = -\gamma_\tau(\bar u,\bar\tau)\bar\tau, \tag{4.10}$$

$$\lambda\,\partial\bar\mu/\partial\lambda = -\tfrac{1}{2}\gamma_\mu(\bar u,\bar\tau)\bar\mu, \tag{4.11}$$

with the initial conditions $\bar u(1)=u$, $\bar\tau(1)=\tau$, $\bar\mu(1)=\mu$.

Before presenting our solutions of the characteristic equations, we note the following limiting cases as $\lambda\to 0$.

$$\tau=0:\qquad \bar u\approx u^*,\qquad (\bar\tau/\tau)\approx\lambda^{-1/\nu},\qquad (\bar\mu/\mu)\approx\lambda^{-\beta/\nu}, \tag{4.12}$$

$$\tau\neq 0, N\neq 1:\qquad \bar u\approx u^{**},\qquad (\bar\tau/\tau)\approx(\bar\mu/\mu)^2\approx\lambda^{-(2-\varepsilon)}, \tag{4.13}$$

$$\tau\neq 0, N=1:\qquad \bar u\approx u\lambda^{-\varepsilon},\qquad (\bar\tau/\tau)\approx(\bar\mu/\mu)^2\approx\lambda^{-2}, \tag{4.14}$$

where

$$Su^*=\frac{6\varepsilon}{(N+8)},\qquad \nu^{-1}=(1-\tfrac{1}{2}\varepsilon)\beta^{-1}=2-\left(\frac{N+2}{N+8}\right)\varepsilon, \tag{4.15}$$

reproduce the standard fixed point coupling constant and critical exponents correct to order ε. Full solutions of (4.9)–(4.11) are difficult to obtain (cf Amit and Goldschmidt 1978). However, one may verify that the expressions

$$\bar u=u^*(3-2t/\lambda^2)^{\varepsilon/2}Q(t/\lambda^2), \tag{4.16}$$

$$\bar\mu^2=M^2\lambda^{-(2-\varepsilon)}(3-2t/\lambda^2)^{\varepsilon/2}Q(t/\lambda^2), \tag{4.17}$$

$$\bar\tau=t\lambda^{-1/\nu}(3-2t/\lambda^2)^{(2\nu-1)/2\nu}Q(t/\lambda^2), \tag{4.18}$$

$$Q(t)=\tfrac{1}{9}(N+8)[1+\tfrac{1}{9}(N-1)(3-2t)^{\varepsilon/2}]^{-1}, \tag{4.19}$$

obtained by moderately inspired guesswork, satisfy the characteristic equations up to corrections which can be regarded as $O(\varepsilon^2)$, and exhibit the required limiting behaviour. In order to simplify these solutions, we have taken two steps to eliminate inessential corrections to the leading scaling behaviour. Firstly, we have set the usual coupling constant

$$\tilde u=u[1-\tfrac{3}{4}Su\ln(3-2\tau)+O(u^2)], \tag{4.20}$$

defined by (2.5), equal to its critical fixed point value u^*. Secondly, we have anticipated the scaling limit $\tau \sim t \to 0$ and $\mu \sim M \to 0$, and retained only the leading terms. For this reason, the solutions do not exactly satisfy the stated initial conditions. Note that at this order, the quantity inside the bracket $(3-2t/\lambda^2)^{\varepsilon/2}$ may be adjusted by corrections of order ε. Such adjustments will not be indicated explicitly in what follows. Also, the values of λ we require will always avoid the singularity at $\lambda^2 = 2t/3$.

We now apply these results to obtain the equation of state in the critical region from the Ward identity

$$H = Mf_{\pi\pi}(0; \tau, \mu, u) \tag{4.21}$$

in which the magnetic field H has been rendered dimensionless by extraction of the appropriate power of κ. Using (4.8) and the perturbation expansion to first order, we find

$$H/\lambda^2 M = \bar{\tau} + \bar{\mu}^2 + \tfrac{1}{12}(N-1)S\bar{u}(\bar{\tau}+\bar{\mu}^2)\ln(\bar{\tau}+\bar{\mu}^2) + \tfrac{1}{4}S\bar{u}(\bar{\tau}+3\bar{\mu}^2)\ln(\bar{\tau}+3\bar{\mu}^2) \\ -\tfrac{1}{4}S\bar{u}(\bar{\tau}+3\bar{\mu}^2)\ln(3-2\bar{\tau}), \tag{4.22}$$

the final term arising from our τ-dependent renormalisation scheme. Owing to our choice of α in (2.16), this equation may be completely exponentiated by choosing the free parameter λ so that

$$\bar{\tau} + \bar{\mu}^2 = 1. \tag{4.23}$$

Then (4.22) reads

$$\lambda^2 \simeq H/M. \tag{4.24}$$

The equation of state is traditionally expressed in terms of the scaling variables

$$y = HM^{-\delta}, \qquad x = tM^{-1/\beta}, \tag{4.25}$$

with

$$(\delta-1)^{-1} = \tfrac{1}{4}(2-\varepsilon) + O(\varepsilon^2).$$

Substitution of (4.24), (4.17) and (4.18) into (4.23) yields an implicit equation of state which to lowest order gives

$$y = \lambda^2/M^2 = 1 + t/M^2 = 1 + x \tag{4.26}$$

or

$$\lambda^2/t = 1 + 1/x. \tag{4.27}$$

Using this lowest-order result, the equation becomes

$$(3+x)^{\varepsilon/2} + 2^{3\varepsilon/(N+8)}x(3+x)^{(N+2)\varepsilon/2(N+8)} \\ = \left(\frac{9}{N+8}\right)y + \left(\frac{N-1}{N+8}\right)y^{1-\varepsilon/2}(3+x)^{\varepsilon/2} \tag{4.28}$$

after some rearrangement, and rescaling of x by $2^{3\varepsilon/(N+8)}$ to ensure $y=0$ when $x=-1$.

This result incorporates in a single equation most of the previously established or conjectured features of the equation of state. Thus the Widom function

$$f(x) = y \tag{4.29}$$

is analytic at $x = 0$, and has the ε expansion

$$f(x) = 1 + x + \frac{\varepsilon}{2(N+8)}[3(3+x)\ln(3+x) + (N-1)(1+x)\ln(1+x) + 3(\ln 2)x] + O(\varepsilon^2), \tag{4.30}$$

which agrees with the result quoted by Wallace (1976) apart from a normalising factor. On the other hand, the quantity $(1+x)$ has the expected form of a double expansion in powers of y and $y^{1-\varepsilon/2}$ near the coexistence curve $x \to -1$, $y \to 0$, with the non-analytic terms vanishing correctly in the Ising-like case $N = 1$. (The essential singularity expected for an Ising-like system (Langer 1967) is, of course, not detected by the ε expansion.) The first two coefficients of this series have been obtained in the ε expansion by Wallace and Zia (1975) and we recover their results to lowest order. In the spherical model limit $N \to \infty$, we have

$$(1+x) = y^{1-\varepsilon/2}, \tag{4.31}$$

in agreement with the exact result of Brézin and Wallace (1973). The one defect of which we are aware is that the function g, defined by

$$y = x^{\gamma} g(x^{-2\beta}), \tag{4.32}$$

with $\gamma = \beta(\delta - 1)$, is not manifestly analytic for large x, although the leading power of x is correct. The required analyticity, which is implicit in the renormalisation group equation (4.4), may be restored in various ways by parametric representations. One possibility, suggested by J Nicoll (private communication) in a slightly different form, is to replace (4.28) by the equation

$$K^a + 2^c x K^b = y\left[\frac{9}{N+8} + \left(\frac{N-1}{N+8}\right)\left(\frac{K}{y}\right)^{\varepsilon/2}\right] \tag{4.33}$$

where

$$a = (\gamma - 2\beta)/\gamma = \varepsilon/2 + O(\varepsilon^2), \tag{4.34}$$

$$b = (\gamma - 1)/\gamma = \tfrac{1}{2}[(N+2)/(N+8)]\varepsilon + O(\varepsilon^2), \tag{4.35}$$

$$c = (1 - 2\beta)/2\beta = [3/(N+8)]\varepsilon + O(\varepsilon^2), \tag{4.36}$$

and K is defined as the solution of

$$K = 3K^a + 2^c x K^b. \tag{4.37}$$

This is equivalent to (4.28) at $O(\varepsilon)$, and has the same analytic properties near $x = -1$ and $x = 0$. For large x, however, one may write

$$K = x^{\gamma} \bar{K} \tag{4.38}$$

and it is then straightforward to show that $\bar{K}$ and g are analytic in $x^{-2\beta}$. With this modification, our result coincides with that of Nicoll and Chang (1978), and is essentially equivalent to that of Schäfer and Horner (1978).

The transverse correlation function, $G_{\pi\pi}(p) = \Gamma_{\pi\pi}(p)^{-1}$, may be found in the same way, with the unremarkable result

$$\Gamma_{\pi\pi}(p) = M^{\gamma/\beta}\left[y + q^2 + \tfrac{1}{3}S\bar{u}\int_0^1 ds\,\ln\left(1 + \frac{s(1-s)q^2}{y+2s}\right) + O(\varepsilon^2)\right], \tag{4.39}$$

where

$$q = |p| M^{-\nu/\beta} \tag{4.40}$$

and

$$\bar{u} = u^* \tfrac{1}{9}(N+8)(2+y)^{\varepsilon/2}[y^{\varepsilon/2} + \tfrac{1}{9}(N-1)(2+y)^{\varepsilon/2}]^{-1}. \tag{4.41}$$

As expected (see e.g. Schäfer and Horner 1978), $\Gamma_{\pi\pi}$ is analytic at $q = 0$ for all y. The critical singularity $\Gamma_{\pi\pi} \sim |p|^{2-\eta}$ is absent at this order in ε. A different technique (Brézin *et al* 1974) is required to exponentiate the remaining logarithm and obtain the correct non-leading behaviour at large q.

The interesting behaviour of the longitudinal two-point function $\Gamma_{\sigma\sigma}$ is conveniently exhibited by use of the Ward identity

$$\Delta\Gamma_{\sigma\sigma}(p) = \Gamma_{\sigma\sigma}(p) - \Gamma_{\pi\pi}(p) = M\Gamma_{\pi\pi\sigma}(0, p, -p). \tag{4.42}$$

The infrared singularity of this quantity may be transferred into a prefactor by the regularising condition

$$\tilde{\mu}^2 + \tilde{\tau} = 1 - p^2/\lambda^2 \tag{4.43}$$

which now replaces (4.23). We obtain

$$\Delta\Gamma_{\sigma\sigma}(p) = M^{\gamma/\beta} P(q, y)[1 + \varepsilon F(q, y) + \mathrm{O}(\varepsilon^2)], \tag{4.44}$$

where F is a lengthy and unilluminating function, while the prefactor P, which arises directly from the renormalisation group equation with (4.43), is given by

$$P(q, y) = \tfrac{2}{9}(N+8)(y+q^2)^{\varepsilon/2}(2+y+q^2)^{\varepsilon/2} \times [(y+q^2)^{\varepsilon/2} + \tfrac{1}{9}(N-1)(2+y+q^2)^{\varepsilon/2}]^{-1}. \tag{4.45}$$

One sees that, for $N \neq 1$, the longitudinal susceptibility near the coexistence curve has the expected form

$$\chi_{\mathrm{L}} \sim P(0, y)^{-1} \sim y^{-\varepsilon/2} \tag{4.46}$$

as $y \to 0$, and that for $y = 0$,

$$\Gamma_{\sigma\sigma}(p) \sim P(q, 0) \sim q^{\varepsilon}. \tag{4.47}$$

For the Ising-like case, $N = 1$, this singularity is removed, and the susceptibility has a finite value at coexistence.

5. Discussion

We have sought to bring together a number of previously established or conjectured features of coexistence in the critical region of isotropic systems, by exhibiting for the first time a single renormalisation scheme for the ϕ^4 model which explicitly regularises both the critical and the Goldstone mode singularities. Our explicit results to first order in ε are displayed in equations (4.28), (4.39) and (4.45) for the equation of state and the correlation functions. The new feature which emerges from our analysis is the existence of a zero-temperature fixed point, with the coupling constant

$$Su^{**} = 6\varepsilon/(N-1) + \mathrm{O}(\varepsilon^2) \tag{5.1}$$

which controls the coexistence behaviour, and is infrared stable in the temperature variable. This result serves to confirm and clarify the assumed equivalence of the ϕ^4 and nonlinear σ models, by exposing the correspondence between their fixed point structures, under appropriate renormalisation groups.

We illustrate this correspondence in a schematic manner suggested by an anonymous referee. The constraint $|\boldsymbol{\phi}_0| =$ constant which leads to the nonlinear σ model may be applied by taking the limit $u_0 \to \infty$, $r_0 \to -\infty$ with (u_0/r_0) fixed. For convenience, define

$$r' = -r_0(1 + r_0^2 + u_0^2)^{-1/2}, \tag{5.2}$$

$$u' = u_0(1 + r_0^2 + u_0^2)^{-1/2}, \tag{5.3}$$

with r_0 and u_0 rendered dimensionless by extraction of appropriate powers of a lattice constant, say. The (r', u') plane may be envisaged as in figure 1. The nonlinear σ model lies on the unit circle, where the polar angle plays the role of temperature. The non-trivial fixed point of Brézin and Zinn-Justin (1976) and the Wilson–Fisher fixed point of the ϕ^4 model, whose precise location depends on the details of the renormalisation group considered, lie on an invariant trajectory corresponding to the critical temperature, which separates the ordered and disordered phases. Presumably, both these fixed points have the same critical exponents, although we are unable to demonstrate this within the approximation schemes which are currently available. As the dimensionality, d, approaches 2, the invariant trajectory moves, and the Wilson–Fisher fixed point should approach the unit circle. For $N > 2$, the two non-trivial fixed points will finally coincide with the coexistence fixed point at $T = 0$, the ordered region disappearing entirely. For $N = 2$, the situation is less clear. An interesting possibility is that, while the fixed point of the nonlinear σ model remains at a finite temperature, the

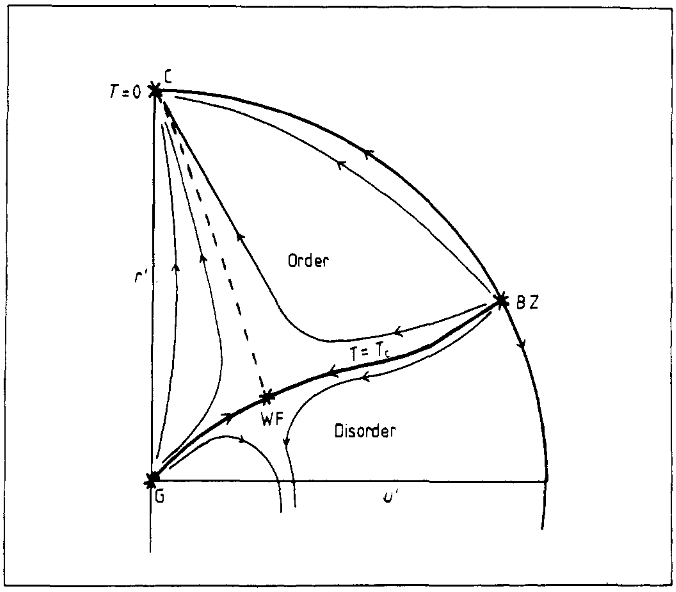

Figure 1. Conjectured renormalisation group trajectories in the (r', u') plane. Fixed points are denoted by G (Gaussian), WF (Wilson–Fisher), BZ (nonlinear σ model: Brézin–Zinn-Justin), C (Coexistence: Brézin–Zinn-Justin and this work).

Wilson–Fisher and coexistence fixed points coincide, the ordered region collapsing to a line of singularities occupying a segment of the unit circle. While this illustration is partly speculative, and figure 1 is probably meaningful only for theories with a finite momentum cut-off, the overall picture seems plausible, and the renormalisation group flows indicated in the figure are consistent with the known stability properties of the various fixed points.

The divergence of (5.1) for the Ising-like case, $N = 1$, may be of some interest. It is apparent from our results that this divergence introduces no anomalous behaviour into the thermodynamic functions from which, indeed, the effects of Goldstone modes are reassuringly absent. It is possible, however, that the presence of this singularity in the renormalisation group structure may have some bearing on the method of analytic continuation to $N = 0$. While this limit has often been taken without obvious difficulty, some questions of interpretation, involving the presence of Goldstone modes, have recently been raised by Moore and Wilson (1980).

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Appendix

We wish to show that the rescaled vertex functions

$$m^{-n}\Gamma_{\sigma^n}(p_1 \ldots p_n), \qquad m^{-n}\Gamma_{\pi\pi\sigma^n}(p_1 \ldots p_{n+2}), \tag{A1}$$

can be rendered finite and non-zero in the limit $m \to \infty$, provided their wavevector arguments are such that they are finite for finite m. For brevity, we refer to the corresponding property of the vertex functions themselves as 'm-finiteness'. Let us first prove the analogous property for the bare vertex functions $\mathring{\Gamma}$ calculated from (2.9). These are free from ultraviolet divergences for $d = 4-\varepsilon < 4$, with poles appearing at $\varepsilon = 0$. That the rescaled functions remain non-zero as $m_0 \to \infty$ is clear, since they contain classes of diagrams which are purely transverse, except for vertices of the form $u_0^{1/2}m_0\pi^2\sigma$, to which the external σ legs are attached. Diagrams of this kind contain only an overall factor of m_0^n. A proof that they also remain finite for $\varepsilon > 0$ will be outlined in the loopwise expansion by induction on the number of loops. We require the Ward identities (see e.g. Amit 1978)

$$(3/u_0)^{1/2}m_0\mathring{\Gamma}_{\pi\pi\sigma^n} = \mathring{\Gamma}_{\sigma^{n+1}} - c_n\mathring{\Gamma}_{\pi\pi\sigma^{n-1}} \tag{A2}$$

with $(3/u_0)^{1/2}m_0 = M$, and purely numerical coefficients c_n, which hold order by order in the loopwise expansion.

Observe first that, since the transverse susceptibility is infinite on the coexistence curve, we have

$$\mathring{\Gamma}_{\pi\pi}(p) = p^2 + \mathrm{O}(p^2u_0m_0^{-\varepsilon}, p^4/m_0^2) \tag{A3}$$

by dimensional analysis, so that $\mathring{\Gamma}_{\pi\pi}$ is m_0-finite. This property is also straightforward to check for all the functions $\mathring{\Gamma}_{\sigma^n}$ and $\mathring{\Gamma}_{\pi\pi\sigma^n}$ at the one-loop level.

Now consider the functions $\mathring{\Gamma}_{\sigma^n}$, and assume that all the vertex functions appearing in (A1) are m_0-finite at the level of L loops. Owing to the unbroken symmetry in the transverse fields, $\mathring{\Gamma}_{\sigma^n}$ contains transverse propagators only in closed loops. Contributions with $L+1$ loops can therefore be decomposed as in figure 2, where diagrams in class (a) are purely longitudinal. In these diagrams, we rescale the integration momenta by a factor of m_0, extracting an overall factor given by dimensional analysis as

$$m_0^{4-n-(n-1+L)\varepsilon}. \tag{A4}$$

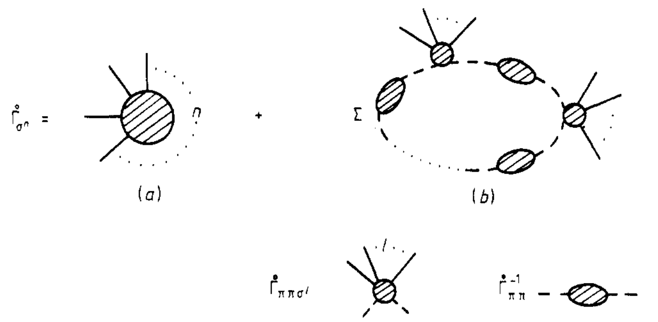

Figure 2. Contributions to the bare longitudinal vertex functions with $(L+1)$ loops. Diagrams of class (a) are purely longitudinal. Those in class (b) contain at least one closed transverse loop, around which are arranged contributions to the indicated vertex functions with no more than L loops.

In the large-m_0 limit, the external momenta now appear in the form $p_i/m_0 \to 0$, but since the longitudinal propagators are massive, no infrared divergences appear. We conclude that this contribution is m_0-finite for all $n > 2$ and $\varepsilon > 0$. This rescaling is inappropriate for the remaining diagrams which contain transverse propagators. However, these diagrams are composed entirely of vertex functions with no more than L loops which, by hypothesis, are m_0-finite. Consequently, the functions $\mathring{\Gamma}_{\sigma^n}$ are all m_0-finite at the level of $(L+1)$ loops. One may now apply (A2) for successive values of n, to show that all the vertex functions $\mathring{\Gamma}_{\pi\pi\sigma^n}$ are m_0-finite at the level of $(L+1)$ loops, which completes the induction.

Consider now a minimal subtraction scheme in which renormalised quantities $\tilde{u}$ and $\tilde{m}$ are defined on the coexistence curve so that $\tilde{m}/\tilde{u}^{1/2}$ in the renormalised Ward identities is proportional to the renormalised magnetisation. The wavefunction renormalisation factor cancels out of (A2), and one would like to conclude that the renormalised vertex functions, which differ from the bare ones only by $\tilde{m}$-independent factors, are also $\tilde{m}$-finite. The obstacle to this conclusion is that, according to (A4), the function $\mathring{\Gamma}_{\sigma\sigma}$ is rendered m_0-finite only by powers of m_0 which are of order ε. Consequently, on expanding in powers of ε and subtracting poles, the $\tilde{m}$-finiteness of $\Gamma_{\sigma\sigma}$ is destroyed by powers of $\ln \tilde{m}$. To overcome this, one need only define a new longitudinal mass, m, by subtracting also the appropriate powers of $\ln(\alpha + m^2)$, and a new coupling constant, u, to remove these logarithms from the renormalised version of (A2). This is equivalent to the scheme described in § 2.

Finally, at the level of two or more loops, one sees that a similar problem arises for $\Gamma_{\pi\pi}$, equation (A3), and is solved by means of an m-dependent wavefunction renormalization factor. According to the discussion of § 3, this should lead to an anomalous dimension exponent $\eta^{**} = 0$ in the coexistence limit.

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